The Edith Cowan University Program

Dr Nathan (Norm) Hoffman, June 2009

This article is in two parts. The first is a statement of principles, based on my experiences, study and reading over a period of fifty years. The second is a description, with illustrative examples, of the Edith Cowan University program that I established in 1992 and have directed since then.

Part 1: Principles that should underpin provisions for mathematically gifted children.

It would clearly be thoughtless to provide for mathematical giftedness by having the students participate in weightlifting courses, because weightlifting does not relate to nor build on the students' special abilities. More appropriately, the planning of a curriculum for mathematically gifted students should be based on considerations of *the nature of mathematics* and *the nature of mathematical giftedness.* What follows are some observations pertaining to each of these two areas.

Mathematics

The following are some observations on the nature of mathematics.

1 Mathematics is a form of intellectual activity. Engaging in mathematical activity is both stimulating and satisfying for many of those who have the intellectual ability to do so.

2 Mathematics is a science. It has been described as the Queen of the sciences, and also as a servant of the sciences. It is a powerful tool with which to increase our understanding of the real world. As a consequence, school mathematics programs should bring out the interrelationships between mathematics and the real world.

3 Mathematics is the natural language of number, order, and form.

4 Mathematics is concerned with pattern and structure, and the use of these to bring order to the study of objects and systems. It is a powerful problem solving tool.

5 Mathematics is concerned with patterns of inference and deduction.

6 Mathematics is a part of our culture and has contributed to the development of Western civilised society.

Mathematical giftedness

The following are some observations on the characteristics of mathematically gifted children.

1 A capacity and willingness to work with abstract notions, eg randomness.

2 An appreciation of the significance of generalisations, eg proofs in geometry.

3 A capacity to perceive patterns and relationships.

4 A capacity to understand patterns of logic and inference, eg mathematical proofs.

5 A capacity to solve multi-stage problems.

The following recommendations are derived from the above two sets of considerations.

Recommendations for programs for mathematically gifted children

1 Provide investigative activities. The students should be given the opportunity to explore problems and develop their own solutions. Often it is possible to structure a sequence of problems, of increasing complexity, leading to a main problem and its solution. Exploration of simple cases is a useful general problem-solving technique.

2 Focus on topics which are mathematically significant. Examples include systematic counting (combinatorics) , chance (random) processes, prime factorisation, and methods of proof.

3 Give preference to topics that are not merely extensions of what is in the standard school program.

4 Give preference to topics which synthesise two or more areas of mathematics. This increases the likelihood that students to will appreciate the inter-relationships between areas of mathematics, rather than seeing mathematics as a collection of unrelated topics.

5 Provide ample time for the exploration of a problem. By this I mean weeks and months, not just one or two hours. (Any problem that I can solve in 10 minutes is not much of a problem for me!)

6 Place more than a little emphasis on the presentation of solutions. Being able to present a solution in a clear, logical way is a most useful communication skill.

Part 2: The Edith Cowan University Program

The Edith Cowan University Program is in its eighteenth year of operation. In that time it has catered for more than 2200 mathematically-able students aged 9 to 15.

The program is year-long. There are five levels within the program: Primary, levels 1 & 2, and Secondary levels 1, 2 & 3. Students attend classes once a week (secondary) or once a fortnight (primary). One measure of the success of the program is that over the years more then 200 students have spent 3 or more years in it.

The secondary program is based on the national program, *Mathematics Challenge for Young Australians*, produced by the Australian Mathematics Trust. Secondary level 1 students participate in the Euler program, level 2 the Gauss program, and level 3 the Noether program. The details of these programs are readily available from the Trust.

The Primary Program

As indicated earlier, the primary program is offered at two levels; Primary 1 (mainly Year 6 students) and Primary 2 (mainly Year 7 students). At each level the program focuses on **systematic approaches to problem solving**. This is done in the context of problems that involve systematic counting (combinatorics), chance (random) processes, and prime factorisation. A consistent theme is that a problem may have no solutions, one solution, several solutions or many solutions. Solving a problem means finding all solutions, not merely one of them (if there is more than one).

In the primary program, the problems are explored over an extended period, typically several months. First a simple version of the problem is examined, then a simple extension, then more complex extensions. Each problem is explored to sufficient depth for students to appreciate the solution strategy involved and the process by which it is obtained. The problems span the range of arithmetic, geometric and symbolic contexts.

The nature of the program is probably best illustrated by some examples of the problems tackled by students. The solutions to the problems, together with some discussion, are given at the end of the article. The reader is urged to attempt the problems before examining the solutions.

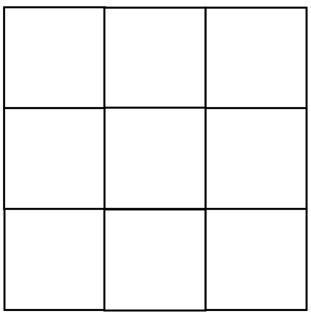
Problem 1: List all the different ways in which $3.60 can be given in exactly eight coins.

Problem 2: 526 is a 3-digit number. The product of its digits is 60 ($5 \times 2 \times 6 = 60$). List all the 3-digit numbers for which the product of the digits is 24.

Problem 3: List all of the different 3-letter arrangements that can be made form the letters of the word “befit”.

Problem 4: List all the different 3-member subsets that can be made from the set consisting of the letters b, e, f, i and t, that is { b , e , f , i , t }

Problem 5: Show all the different ways of selecting two of the nine small squares in the diagram below. Two selections are NOT different if one can be rotated to match the other.



Chance Processes

Another significant strand in the program is concerned with chance processes. Students in Primary level 2 grapple with the following problem, Famous Footballers. It is an adaptation of a problem given by the German mathematician, Arthur Engel[1].

Famous Footballer cards are produced by Kolleggs, the makers of the breakfast cereal, Brekky Bites. There are 100 cards in the whole series. Kolleggs put 5 cards in each large packet of Brekky Bites. You can also buy the cards in packs of 50.

One million cards of each footballer were printed. The whole supply of cards was then thoroughly mixed before the cards were put in the packs and into the packets of Brekky Bites.

Suppose you buy a 50-pack of the cards:

1 Would you expect to get one of each of the 100 different cards? Why?

2 Would you expect to miss out on some footballers? If so, how many footballers?

3 Would you expect to get exactly one card for some footballers? If so, how many footballers?

4 Would you expect to get two cards for some footballers? If so, how many footballers?

5 Would you expect to get three cards for some footballers? If so, how many footballers?

6 Would you expect to get 4 or more of some cards for some footballers? If so, how many footballers?

7 Suppose you open your 50-card pack and count:

- a) how many footballers you have 0 cards for,
- b) how many footballers you have exactly 1 card for,
- c) how many footballers you have exactly 2 cards for,
- d) how many footballers you have exactly 3 cards for,
- e) how many footballers you have 4 or more cards for.

We now make a table showing each student's results. We then calculate the class average for a), b), c), d), and e).
What would be your "best guesses" for the class averages?
On your results sheet write down your "best guess" for these class averages.
Have your results ready to hand in at the next class.

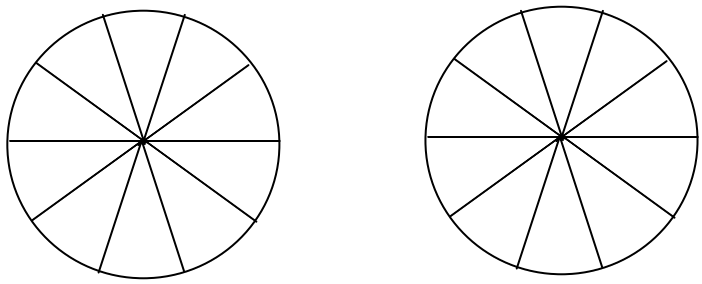
8 How might we systematically explore this problem?

The discussion in the following class brings out the "uncertainty" in the problem. We imagine that we've bought a pack of 50 cards and we want to get some idea as to what the 50 cards might look like if we examine them one at a time, from Card 1 through to Card 50. One solution suggested is to use of a roulette wheel. If we spin and get a result of 52, we would count this as being a card for Footballer 52.

Spinning the wheel 50 times, and noting the number of the footballer at each spin, would give us an understanding of what we might get from a 50-card pack.

But we don’t have a roulette wheel!

We then come to the idea of using two “decimal wheels”. Each is a circle divided into ten congruent sectors:



The sectors in each circle are marked with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. A thumb tack is put through the centre of each circle and a light cardboard spinner (in the shape of a long, thin isosceles triangle) is put on each thumb tack. Then, giving each spinner a push, produces an outcome on each spinner. If the digit on the left wheel is 3 and the digit on the right wheel is 7, we consider this as a card for Footballer 37.

Students quickly realise that there are two problems with this. First, what happens when both spinners show a zero? Second, how can you get a card for Footballer 100? Almost as quickly as these problems are identified, some students realise that problem 1 solves problem 2, that is the double zero is to be counted as a card for Footballer 100.

Each student is given two structured tables. The first is an Outcomes sheet on which the student records the outcome for each of 50 double spins. The second is a Summary sheet onto which the student transposes the 50 outcomes to show how many cards were obtained for each footballer. Each student then produces a table similar to the one below.

Cards	Footballers
0	60
1	31
2	8
3	1
4	0

Cards	Footballers
5	0
6	0
7	0
8	0
9	0

The results are then collected from all students. The table below shows the tabulation of a set of results obtained by 19 students. The heading 3* represents 3 cards for a footballer.

Student	0*	1*	2*	3*	4*	5*
AA	60	31	8	1		
BC	65	25	5	5		
BC	62	27	10	1		
CT	61	30	7	2		
DD	55	40	5			
GD	62	27	10	1		
GM	59	33	7	1		
HA	61	29	9	1		
KJ	59	32	9			
LJ	60	31	8	1		
MC	61	29	9	1		
PD	62	28	8	2		
PP	59	33	7	1		
PS	61	29	9	1		
SA	63	25	11	1		
SM	62	27	10	1		
SV	60	30	10			
SW	56	38	6			
TO	59	32	9			
Average	**60.4**	**30.3**	**8.3**	**1.1**		

The table of results above generates much important discussion. Comparisons with the table below are particularly informative.

The averages of the actual results obtained are quite at variance with the averages of the results predicted by the students (before undertaking the simulation). Students typically underestimate the number of footballers for whom there will be zero cards or one card.

The averages of the actual results obtained are in very close agreement with the theoretical results.

	0*	1*	2*	3*	4*	5*
Predictions	52	22	21	4	2	0
Actual	60.4	30.3	8.3	1.1	0	0
Theoretical	60.5	30.6	7.6	1.2	0.1	0

Drawing on the knowledge and skills developed through the work on systematic counting, arrangements and subsets, the students are able to follow the following derivation of the theoretical result for the expected number of footballers for whom we would get 0 cards from a pack of 50 cards. Bear in mind that these students are only 11 or 12 years old.

Consider a particular footballer, say Footballer 73
The probability that Card 1 is NOT Footballer 73 is 0.99
The probability that NONE of the 50 cards is for Footballer 73 is $(0.99)^{50}$
Using a simple calculator it is easy to show that $(0.99)^{50} = 0.605$ (approx.)
Since there 100 footballers in the series, the expected number of footballers for whom there will be 0 cards in a pack of 50 is $100 \times 0.605 = 60.5$

The students are also shown the derivation of the theoretical results for 1 card and 2 cards.

The simulation solution that the students have used is known as a Monte Carlo Simulation. The significance of this method is borne out by the following quotation from computer scientist, Brian Hayes[2].

> There's also a Monte Carlo problem. I speak not of the Mediterranean principality but of the simulation technique named for that place. The Monte Carlo method got its start in the 1940s at Los Almos, where physicists were struggling to predict the fate of neutrons moving through uranium and other materials. The Monte Carlo approach to this problem is to trace thousands of simulated neutron paths. Whenever a neutron strikes a nucleus, a random number determines the outcome of the event – reflection, adsorption or fission. Today the Monte Carlo method is a major industry not only in physics but also in economics and some areas of the life sciences…

I give this quote in order to support my contention that gifted children should, whenever possible grapple with ***significant*** mathematics, not just run-of-the-mill or "text book" mathematics.

Conclusion

I urge the reader to refer again to the principles stated in Part 1 of this article, and to reflect on the extent to which the examples presented in Part 2 reflect the application of those principles.

References:

1 A. Engel, Teaching Probability in Intermediate Grades, Int. J. Math. Educ. Sci. Technol., Vol 2 (1971), pp 247, 248.

2 B Hayes, Randomness as a Resource, American Scientist, Vol 89, No. 4 (July-August 2001),p 300.

Solutions to the Primary problems

Problem 1: List all the different ways in which \$3.60 can be given in exactly eight coins.

Solution:

\$2	**\$1**	**50c**	**20c**	**10c**	**5c**
1	1	0	2	0	4
1	1	0	1	3	2
1	1	0	0	6	0
1	0	2	2	1	2
1	0	2	1	4	0
1	0	1	5	1	0
0	3	0	2	1	2
0	3	0	1	4	0
0	2	2	2	2	0
0	1	5	0	0	2
0	1	4	3	0	0
0	0	7	0	1	0

Discussion:
Simple though the problem may seem, it is in fact hard to solve. It is relatively easy to find several solutions, but it is difficult to find all twelve solutions. The key is to establish an effective **system**, and then to apply that system carefully and thoroughly. One system is to give the greater preference to retaining the coin of greater value, and only reducing the number of that coin when that is the only option. For example, after the solution:

\$2	**\$1**	**50c**	**20c**	**10c**	**5c**
1	0	2	1	4	0

first consideration is given to reducing the number of 10-cent coins from 4 to 3. When reducing the number of 10-cent coins produces no solution, consideration is given to reducing the number of 20-cent coins from 1 to 0. When this produces no solution, consideration is given to reducing the number of 50-cent coins from 2 to 1. This provides the solution:

\$2	**\$1**	**50c**	**20c**	**10c**	**5c**
1	0	1	5	1	0

When the solution:

\$2	**\$1**	**50c**	**20c**	**10c**	**5c**
0	0	7	0	1	0

is reached, it is clear that there are no more solutions.

Problem 2: 526 is a 3-digit number. The product of its digits is 60 ($5 \times 2 \times 6 = 60$). List all the 3-digit numbers for which the product of the digits is 24.

Solution: The one-digit factorisations of 24 are as follows:

$1 \times 3 \times 8 = 24$
$1 \times 4 \times 6 = 24$
$2 \times 2 \times 6 = 24$
$2 \times 3 \times 4 = 24$

The 3-digit numbers derived from these factorisations are as follows:
$1 \times 3 \times 8 = 24$ gives: 138, 183, 318. 381, 813, 831
$1 \times 4 \times 6 = 24$ gives: 146, 164, 416, 461, 614, 641
$2 \times 2 \times 6 = 24$ gives: 226, 262, 622
$2 \times 3 \times 4 = 24$ gives: 234, 243, 324, 342, 423, 432

Discussion:
Notice the systematic way in which the factorisations have been listed.
Notice the systematic way in which the 3-digit number derived from each factorisation have been listed.

Problem 3: List all of the different 3-letter arrangements that can be made form the letters of the word "befit".

Solution: The 3-letter arrangements are :

bef	bei	bet	bfi	bft	bit	efi	eft	eit	fit
bfe	bie	bte	bif	btf	bti	eif	etf	eti	fti
ebf	ebi	ebt	fbi	fbt	ibt	fei	fet	iet	ift
efb	eib	etb	fib	ftb	itb	fie	fte	ite	itf
fbe	ibe	tbe	ibf	tbf	tbi	ief	tef	tei	tfi
feb	ieb	teb	ifb	tfb	tib	ife	tfe	tie	tif

Discussion: The top row is a systematic listing of the ways in which three of the five letters can be chosen. Each column is then a systematic listing of the arrangements of the three chosen letters at the top of the column.
The systematic listing leads to the development of the algorithm for calculating the number of arrangements.
The number of 3-letter arrangements is $5 \times 4 \times 3 = 60$

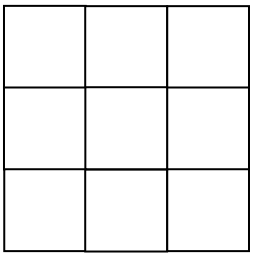
Problem 4: List all the different 3-member subsets that can be made from the set consisting of the letters b, e, f, i and t i.e. { b , e , f , i , t }

Solution: The following are the 3-member subsets of { b , e , f , i , t }

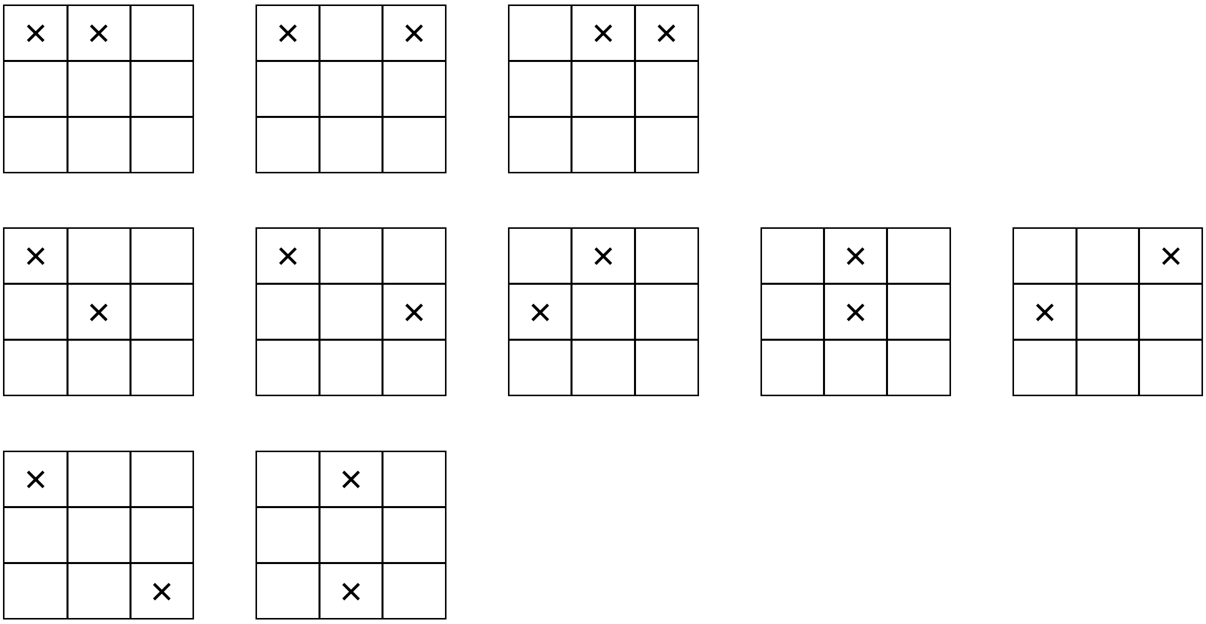
{ b , e , f }	{ e , f , i }	{ f , i , t }
{ b , e , i }	{ e , f , t }	
{ b , e , t }	{ e , i , t }	
{ b , f , i }		
{ b , f, t }		
{ b , i , t }		

Discussion: The relationship between arrangements and subsets is explored. Direct comparison of the results from problems 3 and 4 shows that each 3-member subset has 6 arrangements that are related to it. Problem 3 shows that there are 60 3-letter arrangements, so the number of 3-member subsets is $60 \div 6 = 10$. In more detail the result is (5 × 4 × 3) ÷ (3 × 2 × 1). This is formalised as $^5C_3$, which is read as "Five, choose three".

Problem 5: Show all the different ways of selecting two of the nine small squares in the diagram below. Two selections are NOT different if one can be rotated to match the other.



Solution:

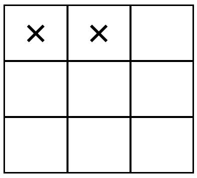


Discussion:
This problem fosters the development of students' spatial concepts.
Simple though it may seem, this is in fact quite a difficult problem.

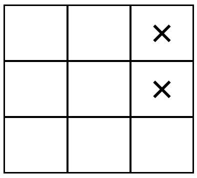
The problem provides an opportunity to link geometric and symbolic aspects of systematic counting. To do this we number the squares as shown below.

1	**2**	**3**
4	**5**	**6**
7	**8**	**9**

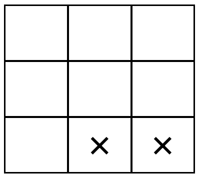
We thus have a set of nine numbers from which we are to select subsets of size two.
The number of subsets of size two is $^9C_2$ = (9 × 8) ÷ (2 × 1) = 36
The first of our selections (shown below) corresponds to { 1 , 2 }.



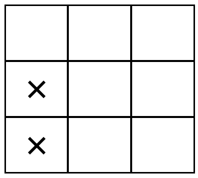
Not only that, but rotation gives other equivalent selections:



{ 3 , 6 }



{ 8 , 9 }

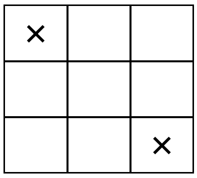


{ 4 , 7 }

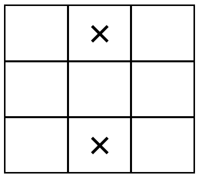
If each selection has four corresponding subsets, as with selection 1 above, then the number of different selections would be 36 ÷ 4 = **9**.

BUT WE ALREADY KNOW THAT THERE ARE **10** DIFFERENT SELECTIONS!

Fortunately there is a simple resolution of this contradiction.



{1 , 9} & {3 , 7}



{2 , 8} & {4 , 6}

These two selections are “special” in that each has only 2 corresponding subsets.
The calculation now becomes:
36 subsets – 4 “special” subsets = 32 “ordinary” subsets
32 “ordinary” subsets ÷ 4 subsets per ”ordinary” selection = 8 ”ordinary” selections
8 ”ordinary” selections + 2 ”special” selections = 10 selections